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TORSIONAL OSCILLATIONS OF A GUN BARREL

by

T. E. Smith Jr.

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DEPARTMENT OF THEORETICAL AND APPLIED MECHANICS
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Proj. Supervisor: M. Stippes

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A Research Project of the
Department of Theoretical and Applied Mechanics
University of Illinois

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This report is concerned with the torsional oscillations in a rifled gun barrel due to an accelerating projectile.

The program is under the technical supervision of Rock Island Arsenal (RIA), Rock Island, Illinois, and the administrative supervision of Chicago Ordnance District.

Respectfully submitted,
University of Illinois

M. Stippes

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Torsional Oscillations of a Gun Barrel

ABSTRACT:

This report is concerned with an analysis of the torsional oscillations of an artillery barrel when subjected to the moment of an accelerating projectile.

A modal series is obtained, the coefficients of which are the time dependent generalized coordinates. For purposes of obtaining numerical results, this series was truncated at two terms; the subsequent system of ordinary differential equations was programmed for the analog computer. Graphic solutions are given for several values of the barrel size to projectile mass ratio.

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NOTATION

x	Coordinate denoting position along the barrel.
r	Coordinate denoting radial distance from the center of the barrel cross section.
ϕ	Angular displacement of barrel cross section and coordinate direction in a cross section perpendicular to the radial direction.
t	Time.
τ	Computer time variable = 4000 t.
ξ	Distance of the projectile along the barrel.
θ	Total angle of rotation of the projectile.
ℓ	Length of the barrel.
G	Shearing modulus for barrel material.
ρ	Mass density of barrel material.
c^2	G/ρ
J	Polar moment of inertia of barrel cross section.
I	Mass moment of inertia of the projectile.
k	Constant determined by the helix angle of the barrel rifling.
a_1, a_2	Torsional spring constants.
$\bar{a}$	An appropriate average angular acceleration of an infinitesimal length of barrel element.
$\gamma_{x\phi}$	Shearing strain between the x and ϕ directions.
$\tau_{x\phi}$	Shearing stress on a cross section.
$M(t)$	Time dependent moment exerted on the barrel.
λ_n	The n^{th} eigenvalue.
ω_n	The n^{th} natural frequency.

$X_n(x)$	The n^{th} normal mode of vibration, or n^{th} eigenfunction.
$q_n(t)$	The n^{th} generlalized coordinate.
$\beta_0(\xi)$	$X_0(\xi)$.
$\beta_1(\xi)$	$X_1(\xi)$.
c^*	$\frac{\rho J}{I}$.

Chapter I

THE EQUATION OF MOTION

1.1 Introduction

The analysis of a complex physical system inevitably necessitates the use of a simplified mathematical model.

The physical dynamics problem to which this investigation has been addressed is the problem of determining the torsional oscillatory behavior of a rifled artillery barrel when subjected to the time dependent moving torque of an accelerating projectile.

The analogous mathematical problem treated here is the problem of determining the angular displacements of a cylindrical shaft when subjected to a time dependent line moment which is dependent in part upon the angular displacements. The shaft is assumed to be made of a homogeneous isotropic material, and the deformations are required to be small so that the material behaves in a linearly elastic manner.

The eigenfunction technique is employed in that solutions are sought each of which can be represented by the product of a normal mode of vibration, a function of distance alone, and a generalized coordinate, a function of time alone. Two systems are considered: that for which the shaft is fixed at one end and otherwise free; and that for which the shaft is free at both ends, but elastically supported at two intermediate points along its length.

The solution of the problem is expressed in the form of an integral equation. For the case of the shaft fixed at one end, the modal series resulting from the use of the eigenfunction technique is truncated at two terms, and the resultant pair of second order, ordinary differential equations with variable coefficients is solved through the use of an analog computer. Numerical results corresponding to some typical artillery data are obtained and presented graphically.

1.2 Derivation of the Equation of Motion

A differential equation of motion for a circular shaft in torsional oscillation about its longitudinal centroidal axis will be derived. The shaft is assumed to be made of an elastic, homogeneous, isotropic material. The angle of twist must be small in order that the proportional limit of the material not be exceeded. The shearing stress on any transverse cross section is assumed to vary directly as the distance from the center of the section; this requires that plane cross sections remain plane and that any straight line segment in a cross section before deformation remain a straight line segment after deformation.

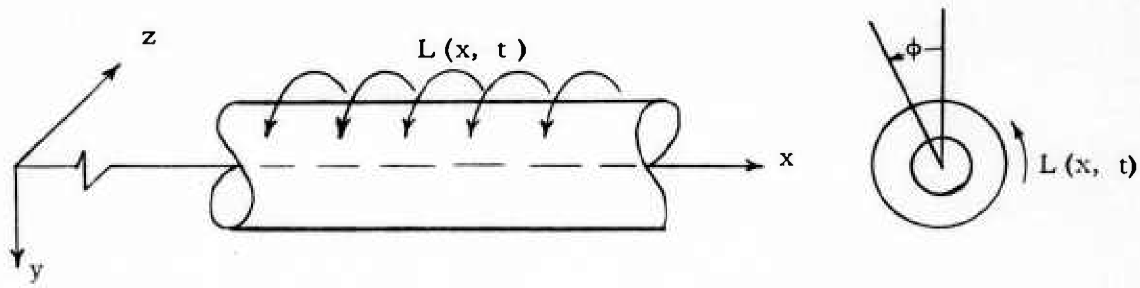

Figure 1

Figure 1 shows a portion of this shaft. $L(x, t)$ is the torque per unit length applied to the shaft, the positive direction taken counterclockwise as shown. $\phi(x, t)$ is the angle through which a particular cross section rotates as a rigid unit. A segment of the shaft Δx units long is shown in Figure 2. $T(x, t)$ is the internal moment exerted on a cross section by the adjoining section.

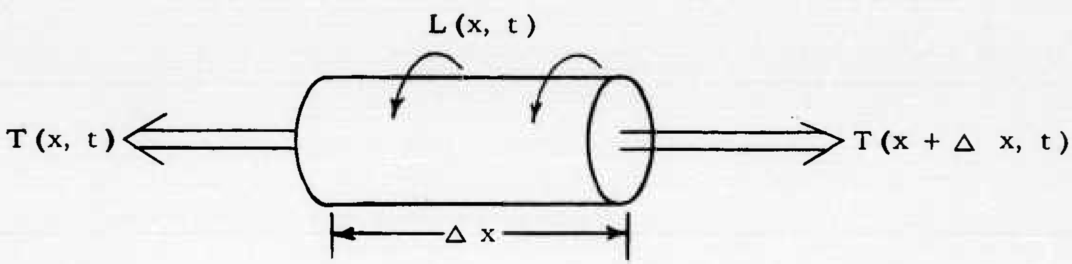

Figure 2

The sum of all moments on the element of Figure 2 must be equal to $I^*\bar{a}$, where I^* is the mass moment of inertia of the element about its centroidal axis and $\bar{a}$ is an appropriate average angular acceleration. Thus,

$$L(x, t)\Delta x + T(x + \Delta x, t) - T(x, t) = I^* \overline{\frac{\partial^2 \phi}{\partial t^2}} . \tag{1}$$

Expanding $T(x + \Delta x, t)$ in a Taylor Series about the point (x, t), we obtain

$$T(x + \Delta x, t) = T(x, t) + \frac{\partial T}{\partial x}(x, t)\Delta x + \ldots O\, \overline{\Delta x}^2 . \tag{2}$$

Substituting (2) into (1) and passing to the limit results in

$$L(x, t) + \frac{\partial T}{\partial x}(x, t) = \rho J \frac{\partial^2 \phi}{\partial t^2}(x, t), \tag{3}$$

where ρ is the mass density of the material and J is the polar moment of inertia of the cross section about the centroidal axis.

The shearing strain, $\gamma_{x\phi}$, is defined to be the angle $(\frac{\pi}{2} - \Omega)$, where Ω is the angle in the deformed state between a line originally parallel to the x-axis and a line in the ϕ-direction.

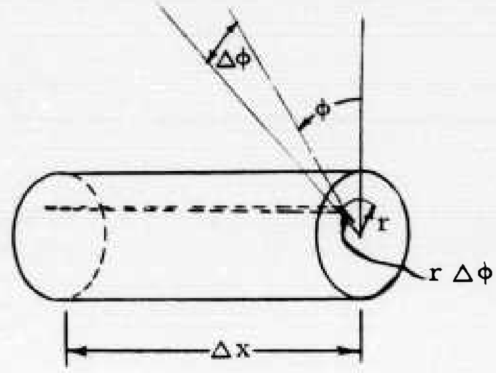

Figure 3

For small deformations $(\frac{\pi}{2} - \Omega)$ is approximately equal to $\tan(\frac{\pi}{2} - \Omega)$. From Figure 3, we see that $\tan(\frac{\pi}{2} - \Omega) = r\,\Delta\phi/\Delta x$. Passing to the limit,

$$\gamma_{x\,\phi} = r\,\frac{\partial\,\phi}{\partial\,x}\,. \tag{4}$$

Substituting the stress-strain relation, $G\,\gamma_{x\,\phi} = \tau_{x\,\theta}$, into Eq. (4), we obtain

$$G\,r\,\frac{\partial\,\phi}{\partial\,x} = \tau_{x\phi}\,. \tag{5}$$

Now the torque exerted on the cross section at x by the adjacent section is $T(x, t)$. This torque is a result of the shearing stress, $\tau_{x\,\phi}$, over the section, so that

$$T(x, t) = \int_A \tau_{x\,\phi}\; r\,dA = J\,G\,\frac{\partial\,\phi}{\partial\,x}\,(x, t). \tag{6}$$

From (6), for a shaft of constant cross sectional area, we see that

$$\frac{\partial\,T}{\partial\,x}\,(x, t) = J\,G\,\frac{\partial^2\,\phi}{\partial\,x^2}\,(x, t)\,. \tag{7}$$

Substituting (7) into (3), we obtain

$$\frac{\partial^2 \phi}{\partial t^2}(x, t) - c^2 \frac{\partial^2 \phi}{\partial x^2}(x, t) = \frac{L(x, t)}{\rho J} , \tag{8}$$

where $c^2 = \frac{G}{\rho}$.

Eq. (8) is the differential equationof motion of an elastic circular shaft to which is applied a time dependent torque, $L(x, t)$, per unit length. It is valid for deformations which are small enough to insure conformity with the behavior assumed.

Chapter II

FORMULATION OF THE PROBLEM AND FORMAL SOLUTION

2.1 We wish to determine the torsional motion of an artillery barrel subjected to the time dependent torque of an accelerating projectile.

Let the barrel itself be represented by the shaft considered in Chapter 1. Its equation of motion then is

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{L(x, t)}{\rho J} . \tag{8}$$

If the moment imparted to the barrel by the projectile is considered to be a line moment and if the distance of the projectile from the left end is designated as $\xi = \xi(t)$, then the following equation may be used to describe the physical system:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{M(t)}{\rho J} \delta(x, \xi), \tag{9}$$

where $\delta(x, \xi)$ is the Dirac Delta function having the property:

$$\int_{x_1}^{x_2} \delta(x, \xi)\, dx = \begin{cases} 0 \text{ for } x_1 \leqq x_2 \leqq \xi \\ 1 \text{ for } x_1 \leqq \xi \leqq x_2 \\ 0 \text{ for } \xi \leqq x_1 \leqq x_2 \end{cases},$$

and $M(t)$ is the time dependent moment upon the shaft.

We now obtain an expression for the moment $M(t)$: The projectile (assumed rigid) at any time t has rotated through a total angle $\theta = \theta(\xi, t)$. This angle is composed of the angle ϕ due to the deformation of the barrel and the angle a due to the helical rifling of the barrel. Hence, the total angle through which the projectile has rotated is

$$\theta = \phi(\xi, t) + k\xi(t) \tag{10}$$

where k is a constant dependent upon the helix angle, so that

$$M(t) = I\ddot{\theta} = I\left[k\ddot{\xi} + \ddot{\phi}(\xi, t)\right] , \tag{11}$$

where I is the mass moment of inertia of the projectile and the dots denote differentiation with respect to time.

Eq. (9) may now be written

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{I}{\rho J}\left[k\ddot{\xi} + \ddot{\phi}(\xi, t)\right] \delta(x, \xi) . \tag{12}$$

2.2 Formulation for Barrel Fixed at One End and Otherwise Free.

Consider Eq. (8) with $L(x, t) = 0$ as applied to Figure 4:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0 . \tag{13}$$

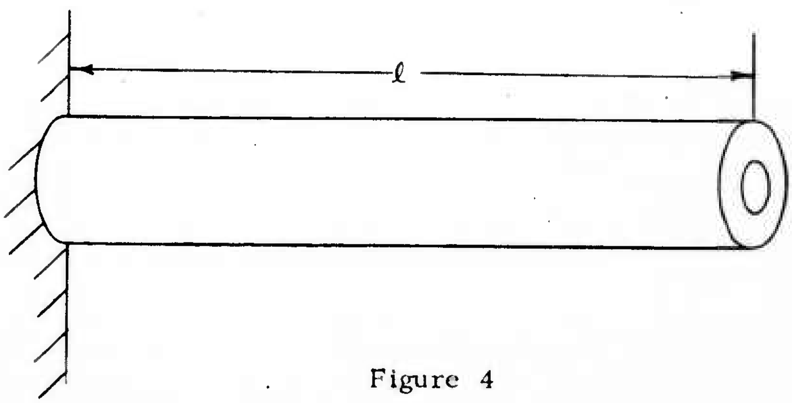

Figure 4

The totality of motions of a continuous vibrating system is known when the eigenvibrations are known. [1] *

We consider free vibrations for which the angular displacement, ϕ , can be expressed as the product of a factor $X(x)$, depending only upon position, and a factor $q(t)$, depending only upon time.

Substituting $\phi = X(x)\,q(t)$ into (13), we obtain

$$\frac{1}{c^2}\frac{\ddot{q}(t)}{q(t)} = \frac{X''(x)}{X(x)}\,, \tag{14}$$

where the primes denote differentiation with respect to x .

The left side of (14) depends only upon time, whereas the right side depends only upon position. Therefore, each must be equal to the same constant $-\lambda^2$, and we obtain from (14)

$$\begin{aligned} X''(x) + \lambda^2 X(x) &= 0 \\ \ddot{q}(t) + \lambda^2 c^2 q(t) &= 0 \;. \end{aligned} \tag{15}$$

Solving the first of (15) ,

$$X(x) = A \sin \lambda x + B \cos \lambda x \;. \tag{16}$$

Now the boundary conditions are

(a) $\phi\,(0,\,t) = 0$

(b) $\dfrac{\partial \phi}{\partial x}\,(\ell\,,\,t) = 0$.

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* The numbers in brackets, unless otherwise designated, refer to the bibliography.

Condition (a) states that there is no displacement at the fixed end. Condition (b) states that the moment at the free end is zero.

From (a) we obtain

$$B = 0.$$

From (b) we see that $A \lambda \cos(\lambda \ell) = 0$. But if $A = 0$, the solution would be trivial. Therefore $\lambda \ell = (2n+1)\frac{\pi}{2}$ $(n = 0, 1, 2, \ldots)$ must be taken as the solution in order that the boundary conditions be fulfilled. Hence the solution to (13) subject to conditions (a) and (b) above is

$$\phi(x, t) = \sum_{n=0}^{\infty} A_n \sin \lambda_n x \, q_n(t), \tag{17}$$

where $\lambda_n = \frac{(2n+1)\pi}{2\ell}$.

The functions $\sin \lambda_n x$ are the eigenfunctions of the differential equation (13) under the boundary conditions $X(0) = 0$ and $X'(\ell) = 0$.

<u>Orthogonality of Eigenfunctions $X_n(x)$</u>

We shall show that the set of eigenfunctions $X_n(x)$ is an orthogonal set over $[0, \ell]$, i.e.,

$$\int_0^\ell X_n X_m \, dx = 0 \qquad \text{for} \quad m \neq n.$$

To this end, we multiply the first of Eqs. (15) by $X_n(x)$, then by $X_m(x)$ and subtract the two results. This yields

$$X_n X''_m - X_m X''_n + (\lambda^2_m - \lambda^2_n) X_n X_m = 0. \tag{18}$$

Integrating (18) over the length ℓ,

$$(\lambda_m^2 - \lambda_n^2) \int_0^{\ell} X_n X_m \, dx + \int_0^{\ell} \frac{d}{dx} (X_n X'_m - X_m X'_n) \, dx = 0 . \quad (19)$$

Since $X_n(0) = X_m(0) = 0$, $X'_m(\ell) = X'_n(\ell) = 0$, and $\lambda_m \neq \lambda_n$, (19) states that

$$\int_0^{\ell} X_m X_n \, dx = 0 \text{ for } n \neq m .$$

Hence the $X_n(x) = \sin \lambda_n x$ subject to the conditions $X_n(0) = 0$ and $X'_n(\ell) = 0$ form an orthogonal set of functions over the interval $[0, \ell]$. This set of functions may be normalized by requiring that

$$\int_0^{\ell} A_n^2 \sin^2 \lambda_n x \, dx = 1 . \quad (20)$$

As a consequence of (20), $A_n = \sqrt{\frac{2}{\ell}}$

The trigonometric functions $\sin \lambda_n x$ form a complete set on the interval $[0, \ell]$, and therefore any piecewise smooth function on that interval may be expanded in terms of $\sin \lambda_n x$ such that the series expansion converges uniformly to the function. (See for example, [1]).

We now show that we can express the Dirac Delta function in terms of the eigenfunctions. Let $y = p(x)$ be a continuous function such that $p(x) = 0$ unless $\xi - \epsilon \leqq x \leqq \xi + \epsilon$ and $p(x) = \text{constant} = \frac{1}{2\epsilon}$ for x in that interval. This situation is depicted in Figure 5.

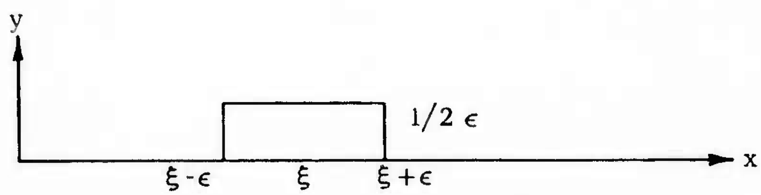


Figure 5

We can expand $p(x)$ in terms of $\sin \lambda_n x$:

$$p(x) = \sum_{n=0}^{\infty} p_n \sin \lambda_n x$$

where $p_n = \frac{2}{\ell} \int_0^{\ell} p(x) \sin \lambda_n x \, dx = \frac{2 \sin \lambda_n \xi \sin \lambda_n \epsilon}{\lambda_n \epsilon \ell}$

Therefore $p(x) = \frac{2}{\ell} \sum_{n=0}^{\infty} \frac{\sin \lambda_n \xi \sin \lambda_n \epsilon \sin \lambda_n x}{\lambda_n \epsilon}$,

and for $\sqrt{\frac{2}{\ell}} \sin \lambda_n x = X_n(x)$,

$$\lim_{\epsilon \to 0} p(x) = \lim_{\epsilon \to 0} \sum_{n=0}^{\infty} X_n(\xi) X_n(x) \frac{\sin \lambda_n \epsilon}{\lambda_n \epsilon} = \delta(x, \xi)$$

$$= \sum_{n=0}^{\infty} X_n(\xi) X_n(x) \tag{21}$$

Utilizing the expressions (17) and (21) in Eq. (9), we obtain

$$\sum_{n=0}^{\infty} \left[X_n(x) \ddot{q}_n - c^2 X''_n(x) q_n \right] = \frac{M(t)}{\rho J} \sum_{n=0}^{\infty} X_n(x) X_n(\xi) . \tag{22}$$

But $X''_n(x) = -\lambda^2_n X_n(x)$, so that (22) becomes

$$\sum_{n=0}^{\infty} \left[\left(\ddot{q}_n + \omega^2_n q_n - \frac{M(t) X_n(\xi)}{\rho J} \right) X_n(x) \right] = 0 , \tag{23}$$

where $\omega^2_n = c^2 \lambda^2_n$.

Because the functions $X_n(x)$ form a complete set on the interval $[0, \ell]$, each of the coefficients in (23) must be zero. Hence,

$$\ddot{q}_n(t) + \omega^2_n q_n(t) = \frac{M(t) X_n(\xi)}{\rho J} \tag{24}$$

must be solved for the $q_n(t)$.

2.3 Formulation for Barrel Free at Both Ends With Two Elastic Torsion Supports Along its Length.

This system is depicted in Figure 6.

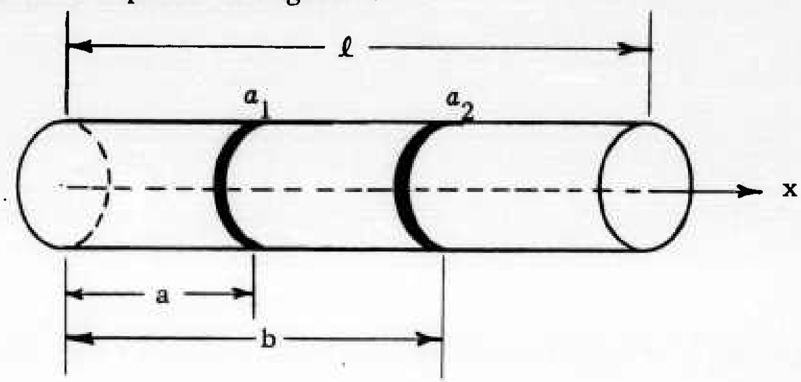


Figure 6

The differential equation for this case is the same as that previously considered, namely

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{M(t)}{\rho J} \delta(x, \xi) . \tag{9}$$

The supports at a and b resist the angular displacement of the barrel such that

$$\begin{aligned} T_1 &= a_1 \phi(a, t) \\ T_2 &= a_2 \phi(b, t) , \end{aligned} \tag{25}$$

where a_1 and a_2 are constants.

The boundary conditions which must be satisfied are: (1) the moments across the supports must be continuous. This condition yields

$$\begin{aligned} J G X'_n(a^-) - J G X'_n(a^+) &= -a_1 X_n(a) \\ J G X'_n(b^-) - J G X'_n(b^+) &= -a_2 X_n(b) ; \end{aligned} \tag{26}$$

(2) ϕ must be continuous at a and b. Therefore

$$X_n(a^-) = X_n(a^+)$$
$$X_n(b^-) = X_n(b^+) \; ; \tag{27}$$

(3) the ends must be without moment so that,

$$X'_n(0) = X'_n(\ell) = 0 \, . \tag{28}$$

We know that $X_n(x)$ must satisfy, in addition to the six boundary conditions given by (26), (27), and (28),

$$X''_n(x) + \lambda^2_n X_n(x) = 0$$

in each of the three regions. The differential equation can be satisfied by the functions $X_n(x)$, defined, for convenience, by

$$X_n(x) = \mu_n(x) = A_n \cos \lambda_n (a-x) + B_n \sin \lambda_n (a-x) \quad \text{for } 0 \leqq x \leqq a \tag{29}$$

$$X_n(x) = \eta_n(x) = C_n \cos \lambda_n x + D_n \sin \lambda_n x \quad \text{for } a \leqq x \leqq b \tag{30}$$

$$X_n(x) = \zeta_n(x) = E_n \cos \lambda_n (x-b) + F_n \sin \lambda_n (x-b) \quad \text{for } b \leqq x \leqq \ell \, . \tag{31}$$

By (26) and (27) we find that

$$J G \lambda_n B_n + J G \lambda_n (D_n \cos \lambda_n a - C_n \sin \lambda_n a) = A_n \alpha_1$$
$$= \alpha_1 (C_n \cos \lambda_n a + D_n \sin \lambda_n a) \tag{32}$$

and

$$-J\,G\,\lambda_n F_n + J\,G\,\lambda_n (D_n \cos\lambda_n b - C_n \sin\lambda_n b) = -E_n a_2$$

$$= -a_2 (C_n \cos\lambda_n b + D_n \sin\lambda_n b)\ , \tag{33}$$

so that

$$A_n = C_n \cos\lambda_n a + D_n \sin\lambda_n a \tag{34}$$

$$E_n = C_n \cos\lambda_n b + D_n \sin\lambda_n b \tag{35}$$

$$B_n = \frac{A_n a_1}{J\,G\,\lambda_n} + C_n \sin\lambda_n a - D_n \cos\lambda_n a \tag{36}$$

$$F_n = \frac{+E_n a_2}{J\,G\,\lambda_n} - C_n \sin\lambda_n b + D_n \cos\lambda_n b\ . \tag{37}$$

By condition (28) we have

$$A_n \sin(\lambda_n a) - B_n \cos(\lambda_n a) = 0 \tag{38}$$

$$-E_n \sin\lambda_n (\ell - b) + F_n \cos\lambda_n (\ell - b) = 0\ . \tag{39}$$

Substituting expressions (34), (35), (36) and (37) into (38) and (39) we obtain two simultaneous, homogeneous, algebraic equations in C_n and D_n. In order that the equations have a non-trivial solution, the determinant of the coefficients must be equal to zero. This condition produces the following equation in λ_n:

$$(J\,G)^2 \lambda_n^2 \sin\lambda_n \ell + J\,G\,\lambda_n \left[a_1 \cos\lambda_n a \cos\lambda_n (\ell - a)\right.$$

$$\left. + a_2 \cos\lambda_n b \cos\lambda_n (\ell - b)\right] + a_1 a_2 \cos\lambda_n a \cos\lambda_n (\ell - b) \sin\lambda_n (b-a)$$

$$= 0 \tag{40}$$

The roots of (40) are the eigenvalues for the problem of the barrel having two elastic torsion supports along its length.

The eigenvalues having been obtained by Eq. (40), we solve the homogeneous equations in the following form:

$$C_n = c_n \psi_n$$

$$D_n = d_n \psi_n \quad .$$

We are at liberty to choose one of the c_n or d_n equal to unity and then determine the other by either of Eqs. (38) or (39). The ψ_n however, are completely arbitrary. We may, therefore, choose them so that the functions $X_n(x)$ are normalized, i. e., so that

$$\int_0^{\ell} X_n^2(x)\, dx = 1 \,. \tag{41}$$

By the method of Rayleigh [2], Eq. (41) may be written

$$\int_0^{\ell} X_n^2(x)\, dx = \frac{1}{2}\left[x\left(\frac{X'_n}{\lambda_n}\right)^2 + x\, X_n^2 - \frac{1}{\lambda_n^2}\, X_n X'_n \right]_0^{\ell} = 1 \,. \tag{42}$$

Therefore, for the problem at hand, Eq. (42) demands that the ψ_n be chosen such that

$$\begin{aligned} & a\left[\mu'^2_n(a) - \eta'^2_n(a)\right] + b\left[\eta'^2_n(b) - \zeta'^2_n(b)\right] \\ & + \lambda_n^2\, \ell\, \zeta_n^2(\ell) + \frac{a_1}{JG}\eta_n^2(a) + \frac{a_2}{JG}\eta_n^2(b) = 2\lambda_n^2 \end{aligned} \tag{43}$$

We seek to show that the set of functions $X_n(x)$ defined by Eqs. (29), (30), and (31) is orthogonal. To this end we write the differential equation for two individual terms of the set:

$$X''_n(x) + \lambda^2_n X_n(x) = 0$$
$$X''_m(x) + \lambda^2_m X_m(x) = 0 \qquad (44)$$

If we multiply the first of (44) by $X_m(x)$ and the second by $X_n(x)$, subtract the two, and integrate the difference from 0 to ℓ, we obtain

$$(\lambda^2_m - \lambda^2_n) \int_0^{\ell} X_m(x)\, X_n(x)\, dx + \Big[X_n X'_m - X_m X'_n \Big]_0^{\ell} = 0 \qquad (45)$$

Expanding the second term of (45), we obtain by definition (29), (30), and (31),

$$\begin{aligned} \Big[X_n X'_m - X_m X'_n \Big]_0^{\ell} &= \mu_n(a)\,\mu'_m(a) - \mu_m(a)\,\mu'_n(a) - \mu_n(0)\,\mu'_m(0) \\ &\quad + \mu_m(0)\,\mu'_n(0) \\ &\quad + \eta_n(b)\,\eta'_m(b) - \eta_m(b)\,\eta'_n(b) - \eta_n(a)\,\eta'_m(a) + \eta_m(a)\,\eta'_n(a) \\ &\quad + \zeta_n(\ell)\,\zeta'_m(\ell) - \zeta_m(\ell)\,\zeta'_n(\ell) - \zeta_n(b)\,\zeta'_m(b) + \zeta_m(b)\,\zeta'_n(b). \end{aligned} \qquad (46)$$

By (28),

$$\mu'_n(0) = \zeta'_n(\ell) = 0 \qquad \text{for all } n. \qquad (47)$$

By (27),

$$\mu_n(a) = \eta_n(a) \quad \text{and} \quad \eta_n(b) = \zeta_n(b) \quad \text{for all } n. \qquad (48)$$

Hence

$$\begin{aligned} \Big[X_n X'_m - X_m X'_n \Big]_0^{\ell} &= \mu_n(a) \Big[\mu'_m(a) - \eta'_m(a) \Big] \\ &\quad - \mu_m(a) \Big[\mu'_n(a) - \eta'_n(a) \Big] + \eta_n(b) \Big[\eta'_m(b) - \zeta'_m(b) \Big] \\ &\quad - \eta_m(b) \Big[\eta'_n(b) - \zeta'_n(b) \Big]. \end{aligned} \qquad (49)$$

By condition (26)

$$\mu'_m(a) - \eta'_m(a) = \frac{-a_1}{JG}\,\mu_m(a)$$

and

$$\eta'_m(b) - \zeta'_m(b) = \frac{-a_2}{JG}\,\eta_m(b) \quad \text{for all } m\,. \tag{50}$$

Therefore the second term in (45) is zero, and, since λ_n in general is different from λ_m for $m \neq n$,

$$\int_0^{\ell} X_m(x)\, X_n(x)\, dx = 0 \qquad \text{for } m \neq n\,.$$

Hence the $X_n(x)$ defined by (29) through (31) subject to (26) through (28) form an orthogonal set of functions on the interval $[0, \ell]$.

The formal solution to Eq. (24), to which both systems previously discussed reduce, is given by Duhamel's integral ,

$$q_n(t) = \frac{I}{\rho J\,\omega_n}\int_0^{t} X_n[\xi(t')]\left(k\ddot{\xi}(t') + \ddot{\phi}[\xi(t'), t']\right)\sin\omega_n(t-t')\,dt' + a_n\sin\omega_n t + b_n\cos\omega_n t\,. \tag{51}$$

If the projectile begins from rest at the left end of the barrel, then $a_n = b_n = 0$ in Eq. (51). If we multiply (51) by $X_n(\xi)$ and sum over n, we obtain the expression for the deflection at the line moment:

$$\sum_{n=0}^{\infty} X_n(\xi)\, q_n(t) = \frac{I}{\rho J}\int_0^{t}\left(k\ddot{\xi}(t') + \ddot{\phi}[\xi(t'), t']\right)\sum_{n=0}^{\infty}\left(\frac{X_n[\xi(t)]\; X_n[\xi(t')]\,\sin\omega_n(t-t')}{\omega_n}\right)dt'$$
$$= \phi(\xi, t) \equiv \phi_0(t) \tag{52}$$

We may reduce the integrodifferential equation (52) to an integral equation by noting that

$$\phi_0(t) = \int_0^t (t - t')\,\ddot{\phi}_0(t')\,dt'. \tag{53}$$

Relation (53) may be verified by direct integration:

$$\int_0^t (t - t')\,\ddot{\phi}_0(t')\,dt' = -t\,\dot{\phi}_0(0) + \phi_o(t) - \phi_0(0)\,. \tag{54}$$

By the condition that $\phi(x, 0) = \dot{\phi}(x, 0) = 0$, (54) reduces to $\phi_0(t)$.

Substituting (53) into (52), we obtain

$$\int_0^t \left[k\,\ddot{\xi}(t')\,K(t, t') - (t - t')\right] dt' = \int_0^t \left[(t - t') - K(t, t')\right] \ddot{\phi}_0(t')\,dt' \tag{55}$$

where

$$K(t, t') = \frac{1}{\rho J} \sum_{n=0}^{\infty} \frac{X_n\left[\xi(t)\right] X_n\left[\xi(t')\right]}{\omega_n} \sin \omega_n (t - t')\,.$$

The solution of (55) presents formidable mathematical difficulties. An approximate technique will therefore be employed in order to gain some knowledge of the barrel behavior.

Chapter III

APPROXIMATE SOLUTION AND NUMERICAL RESULTS

By taking only the first few terms in the series given by Eq. (17), the solution of (24) can be reduced to that of a system of simultaneous ordinary differential equations. With this end in view, let

$$\phi(x, t) = X_0(x)\, q_0(t) + X_1(x)\, q_1(t) \tag{56}$$

Then Eqs. (24) become:

$$\begin{aligned} &\ddot{q}_0 (c^* - \beta_0^2) - \ddot{q}_1 \beta_0 \beta_1 - 2 \dot{q}_0 \beta_0 \dot{\beta}_0 - 2 \dot{q}_1 \beta_0 \dot{\beta}_1 \\ &-q_0 \beta_0 \ddot{\beta}_0 + c^* \omega_0^2 q_0 - q_1 \beta_0 \ddot{\beta}_1 \doteq k \beta_0 \cdot \ddot{\xi} \end{aligned} \tag{57}$$

and

$$\begin{aligned} &\ddot{q}_1 (c^* - \beta_1^2) - \ddot{q}_0 \beta_1 \beta_0 - 2 \dot{q}_1 \beta_1 \dot{\beta}_1 - 2 \dot{q}_0 \beta_1 \dot{\beta}_0 \\ &- q_1 \beta_1 \ddot{\beta}_1 + c^* \omega_1^2 q_1 - q_0 \beta_1 \ddot{\beta}_0 = k \beta_1 \ddot{\xi} . \end{aligned} \tag{58}$$

Multiply (57) by $(c^* - \beta_1^2)$ and (58) by $\beta_0 \beta_1$ and add to obtain

$$\begin{aligned} &\ddot{q}_0 (c^* - \beta_0^2 - \beta_1^2) = 2 \dot{q}_0 \beta_0 \dot{\beta}_0 + 2 \dot{q}_1 \beta_0 \dot{\beta}_1 \\ &-(c^* \omega_0^2 - \beta_0 \ddot{\beta}_0 - \omega_0^2 \beta_1^2) q_0 - (\beta_0 \beta_1 \omega_1^2 - \beta_0 \ddot{\beta}_1) q_1 + k \beta_0 \ddot{\xi} . \end{aligned} \tag{59}$$

Similarly, multiply (57) by $\beta_0 \beta_1$ and (58) by $(c^* - \beta_0^2)$ and add the two to get

$$\ddot{q}_1 (c^* - \beta_0^2 - \beta_1^2) = 2\dot{q}_1 \beta_1 \dot{\beta}_1 + 2\dot{q}_0 \beta_1 \dot{\beta}_0$$

$$- (c^* \omega_1^2 - \beta_1 \ddot{\beta}_1 - \omega_1^2 \beta_0^2) q_1 - (\beta_0 \beta_1 \omega_0^2 - \beta_1 \ddot{\beta}_0) q_0 + k \beta_1 \ddot{\xi}, \tag{60}$$

where

$$c^* = \frac{\rho J}{I} \quad \text{and} \quad \beta_n = X_n(\xi) .$$

Eqs. (51) and (52) can therefore be solved for the two generalized coordinates, $q_0(t)$ and $q_1(t)$.

The solutions to these equations were obtained in graphic form by means of an analog computer. By comparing orders of magnitude of the various terms in equations (59) and (60) and due to the limitations of the computer, all terms involving derivatives of the β's were neglected in these equations. Also the term $(\beta_0^2 + \beta_1^2)$ was neglected as compared with c^*. With these simplifications, Eqs. (59) and (60) become respectively

$$\ddot{q}_0 = \frac{\omega_0^2 \beta_1^2}{c^*} q_0 - \omega_0^2 q_0 - \frac{\beta_0 \beta_1 \omega_1^2}{c^*} q_1 + \frac{k \beta_0}{c^*} \ddot{\xi} \tag{61}$$

$$\ddot{q}_1 = \frac{\omega_1^2 \beta_0^2}{c^*} q_1 - \omega_1^2 q_1 - \frac{\beta_0 \beta_1 \omega_0^2}{c^*} q_0 + \frac{k \beta_1}{c^*} \ddot{\xi} . \tag{62}$$

It should be noted that the above simplifications have been made primarily due to computer limitations; some of the terms neglected are comparable in magnitude to some retained.

Numerical Results

Some typical artillery data used in determining the coefficients in Eqs. (61) and (62) are given in Table I. The calculated values of the problem parameters are tabulated in Table II.

TABLE I

Typical Artillery Data

Weight of gun tube	1058 lb
Length of tube	7.75 ft
Projectile travel	6.8 ft
Weight of standard H. E. round	33 lbs
Inner tube diameter	4.13 inches
Le Duc's Velocity Formula	$V = \frac{1784\,\xi}{1.0277 + \xi}$
Maximum muzzle velocity for zone 7	1550 ft/sec
Helix angle	$8^{o}\ 56'$

TABLE II

Problem Parameters

$\lambda_0 = \pi/2\,\ell = 0.231\ \mathrm{ft}^{-1}$

$\lambda_1 = \frac{3\ \pi}{2\ \ell} = 0.693\ \mathrm{ft}^{-1}$

G (steel) = 12×10^6 psi

ρ (steel) = 488.8 lbm/ft^3

$\omega_0 = \sqrt{\frac{G}{\rho}}\ \lambda_0 = 2465$ rad/sec

$\omega_1 = \sqrt{\frac{G}{\rho}}\ \lambda_1 = 7394$ rad/sec

$k = \frac{2 \tan 8^{o}\ 56'}{.344\ \mathrm{ft}}$ rad/ft

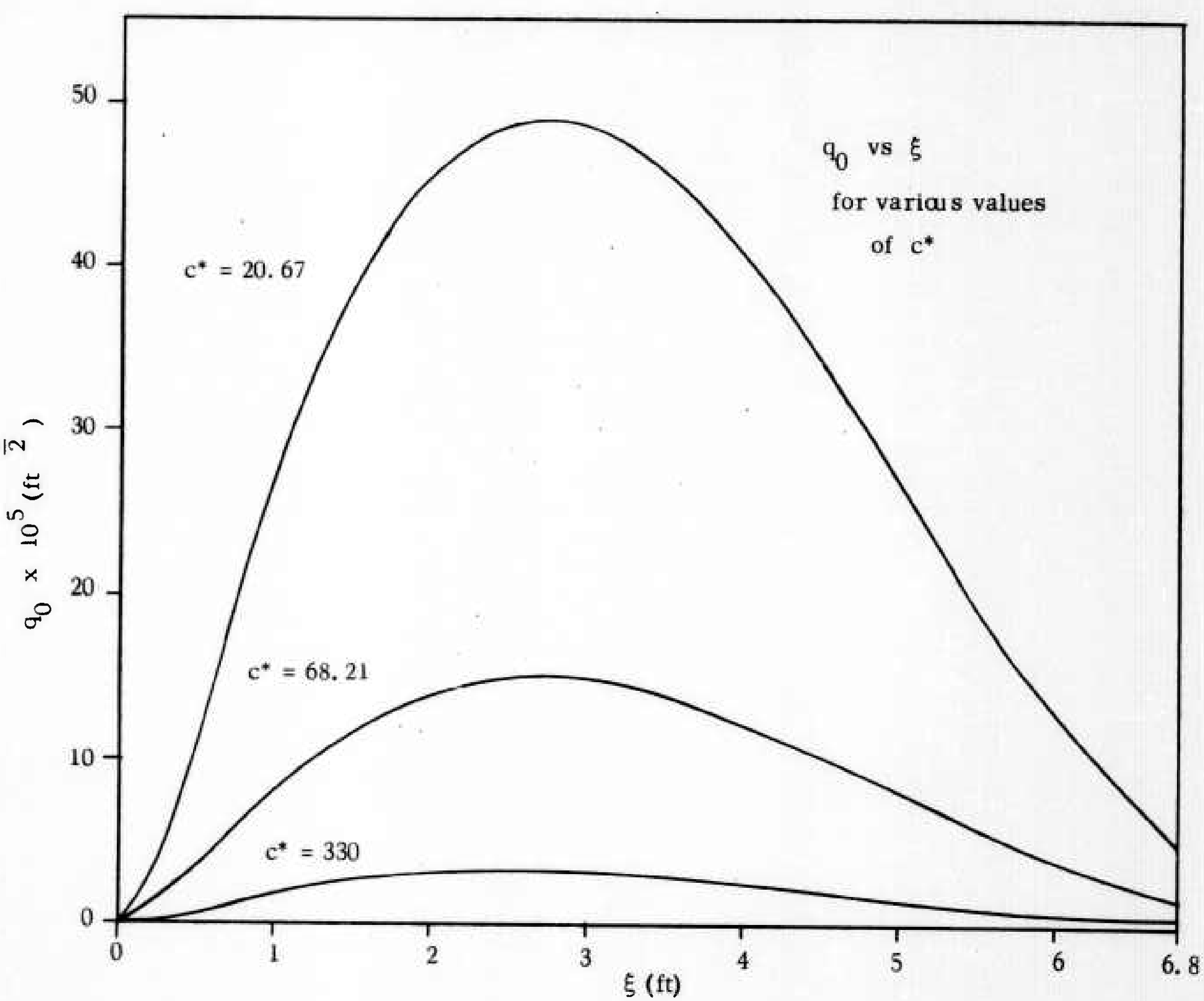
$\beta_0 = \sqrt{\frac{2}{\ell}}\ \sin \lambda_0 \xi = .54 \sin \lambda_0 \xi\ \mathrm{ft}^{-\frac{1}{2}}$

$\beta_1 = \sqrt{\frac{2}{\ell}}\ \sin \lambda_1 \xi = .54 \sin \lambda_1 \xi\ \mathrm{ft}^{-\frac{1}{2}}$

c* (for shell weight of 33 lbs.) = 20.67 ft^{-1}

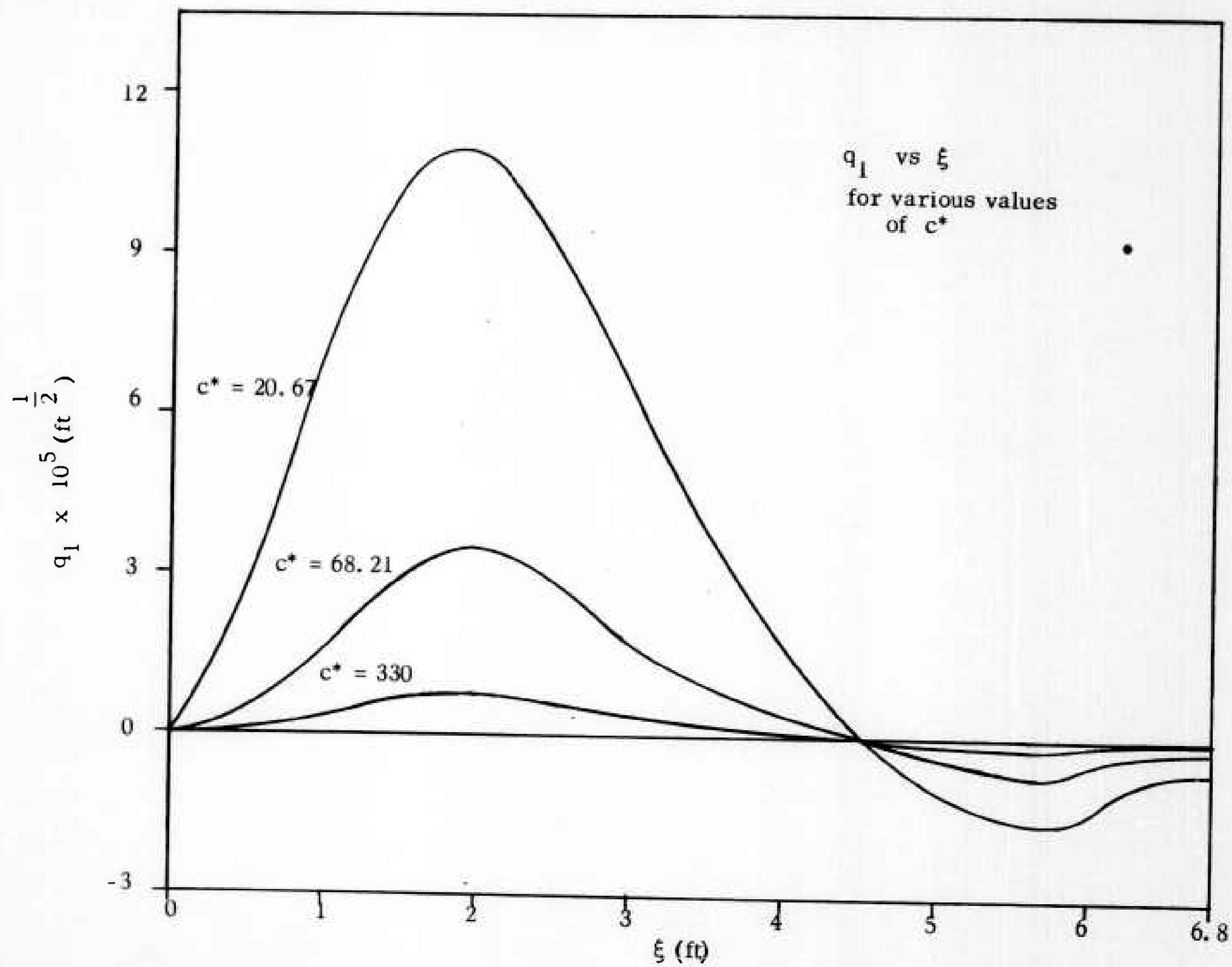
In calculating the value of the polar moment of inertia of the tube J, the tube weight and the length of 7.75 ft were used; in all other calculations, however, the tube length and the projectile travel were taken to be the same value of 6.8 ft.

Figures 7 through 10 show the variations of q_0 and q_1 with time and distance of the projectile down the barrel for several different values of the parameter c^*. In Figure 11 is given the variations of ξ, $\dot{\xi}$ and $\ddot{\xi}$ with time. Figures 12 and 13 show the variation of the q's and $\dot{q}$'s with c^* at the instant the projectile leaves the barrel. We note that the angular displacement and the angular velocity of any section of the barrel are relatively small at this time. In Figure 14 is plotted the angular displacement ϕ (x, t) as a function of x for the nominal value of c^* equal to 20.67 and for three values of time: t_0, the time at which the projectile leaves the barrel; $0.6\ t_0$, the approximate time at which the sum $q_0 + q_1$ reaches a maximum; and $0.92\ t_0$, the time at which q_1 reaches a minimum.



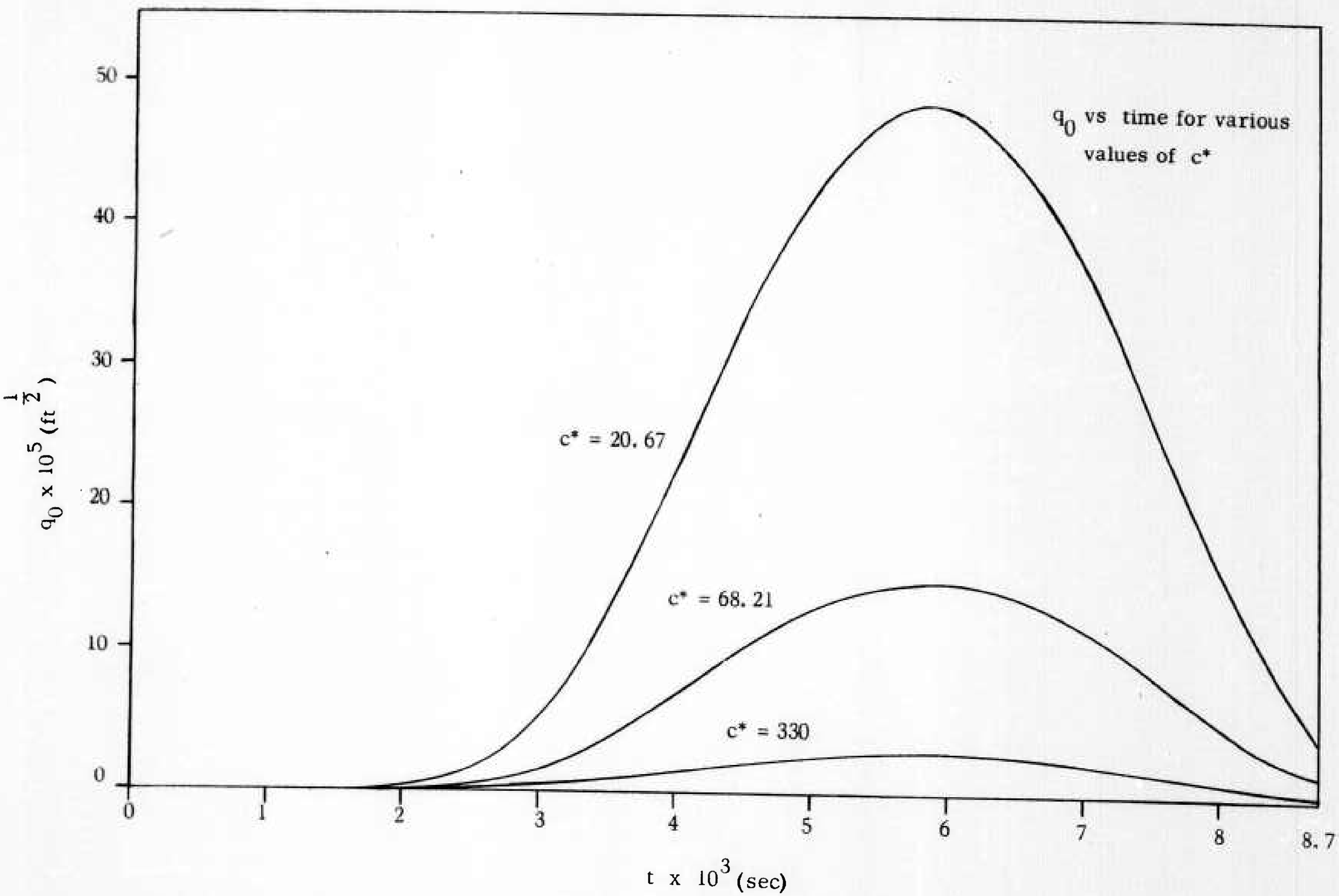
q_0 vs ξ
for various values
of c*
c* = 20.67
c* = 68.21
c* = 330
50
40
30
20
10
0
$q_0 \times 10^5 (ft^{\frac{1}{2}})$
0
1
2
3
4
5
6
6.8
ξ (ft)

Figure 7



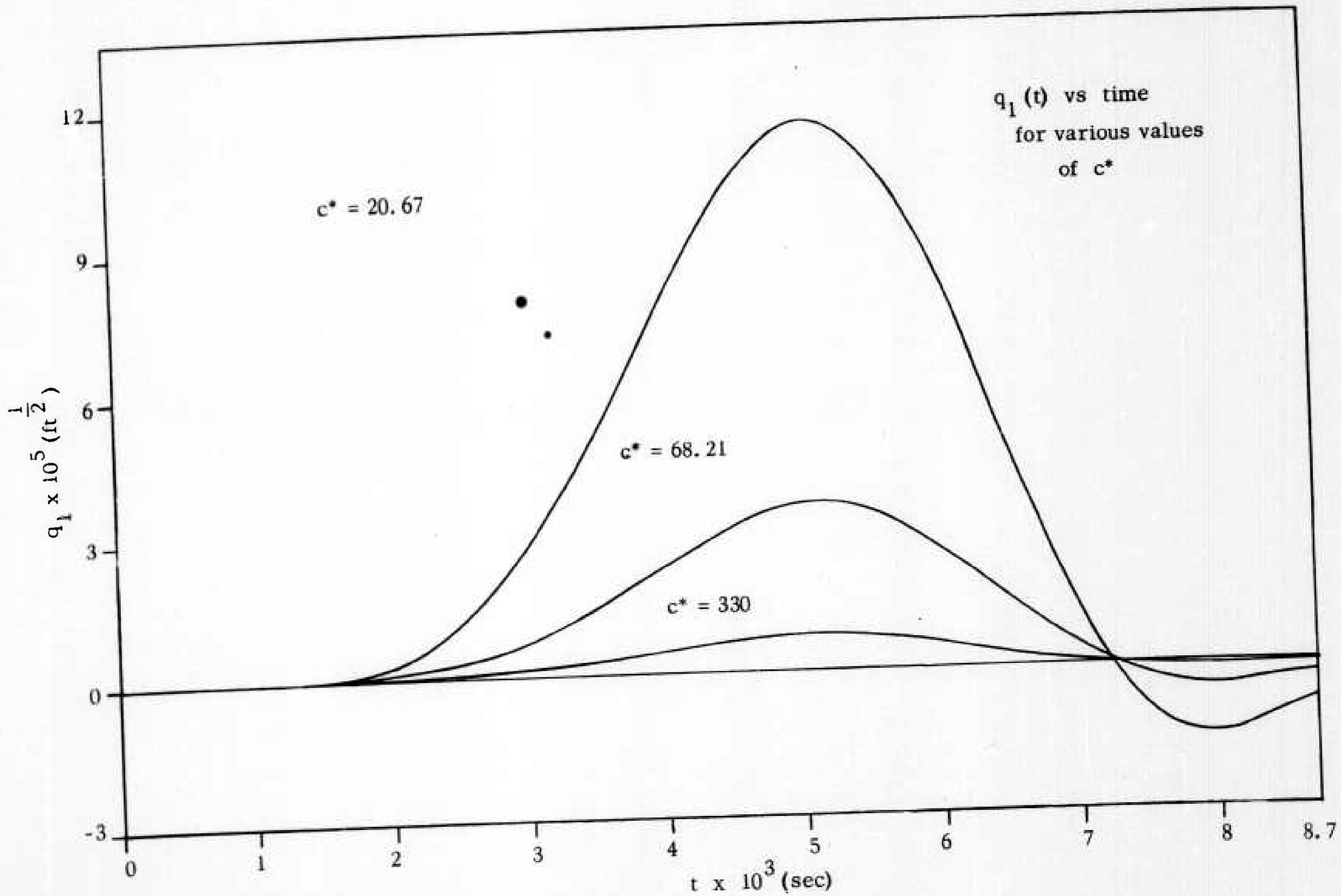
q_1 vs ξ
for various values
of c^*
$c^* = 20.67$
$c^* = 68.21$
$c^* = 330$
$q_1 \times 10^5 (ft^{\frac{1}{2}})$
12
9
6
3
0
-3
0
1
2
3
4
5
6
6.8
ξ (ft)

Figure 8



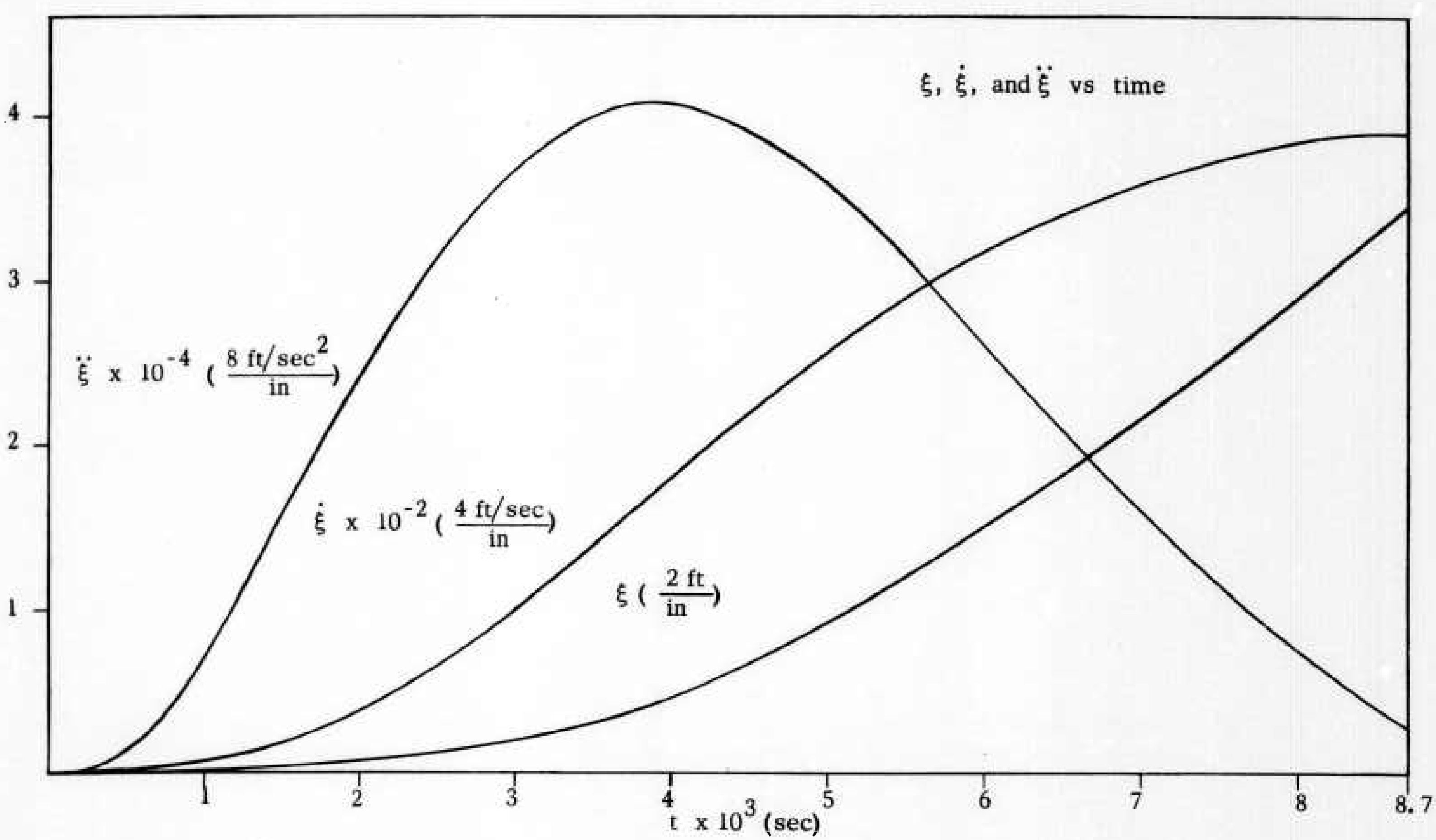
q_0 vs time for various values of c*
c* = 20.67
c* = 68.21
c* = 330
$q_0 \times 10^5$ ($ft^{\frac{1}{2}}$)
50
40
30
20
10
0
0
1
2
3
4
5
6
7
8
8.7
$t \times 10^3$ (sec)

Figure 9



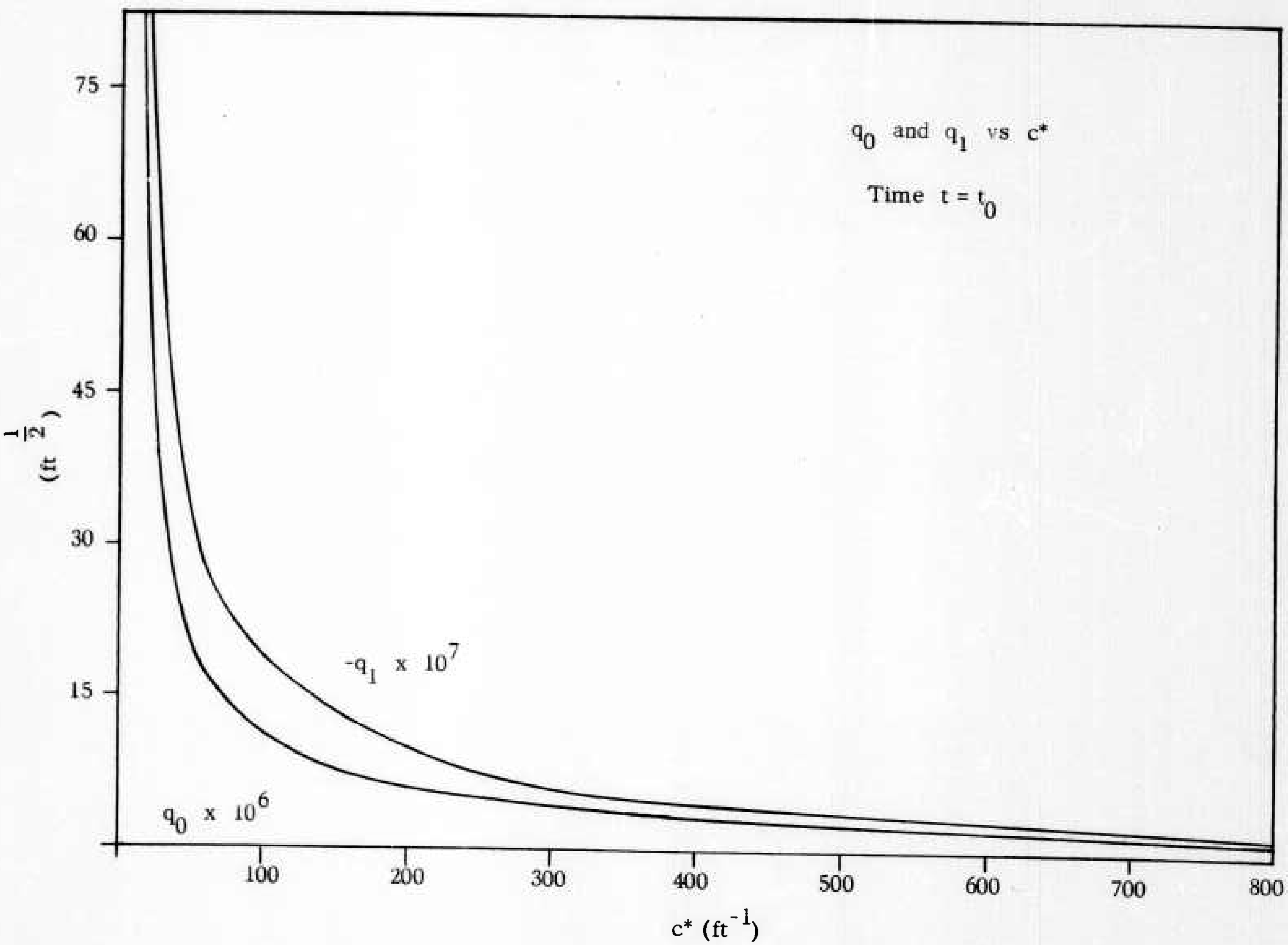
q1 (t) vs time
for various values
of c*
c* = 20.67
c* = 68.21
c* = 330
12
9
6
3
0
-3
q1 x 10^5 (ft^1/2)
0
1
2
3
4
5
6
7
8
8.7
t x 10^3 (sec)

Figure 10



ξ, $\dot{\xi}$, and $\ddot{\xi}$ vs time
$\ddot{\xi} \times 10^{-4} \left(\frac{8\ ft/sec^2}{in}\right)$
$\dot{\xi} \times 10^{-2} \left(\frac{4\ ft/sec}{in}\right)$
$\xi \left(\frac{2\ ft}{in}\right)$
1
2
3
4
5
6
7
8
8.7
$t \times 10^3$ (sec)

Figure 11



q_0 and q_1 vs c^*
Time $t = t_0$
75
60
45
30
15
$(ft^{\frac{1}{2}})$
$-q_1 \times 10^7$
$q_0 \times 10^6$
100
200
300
400
500
600
700
800
c^* (ft^{-1})

Figure 12

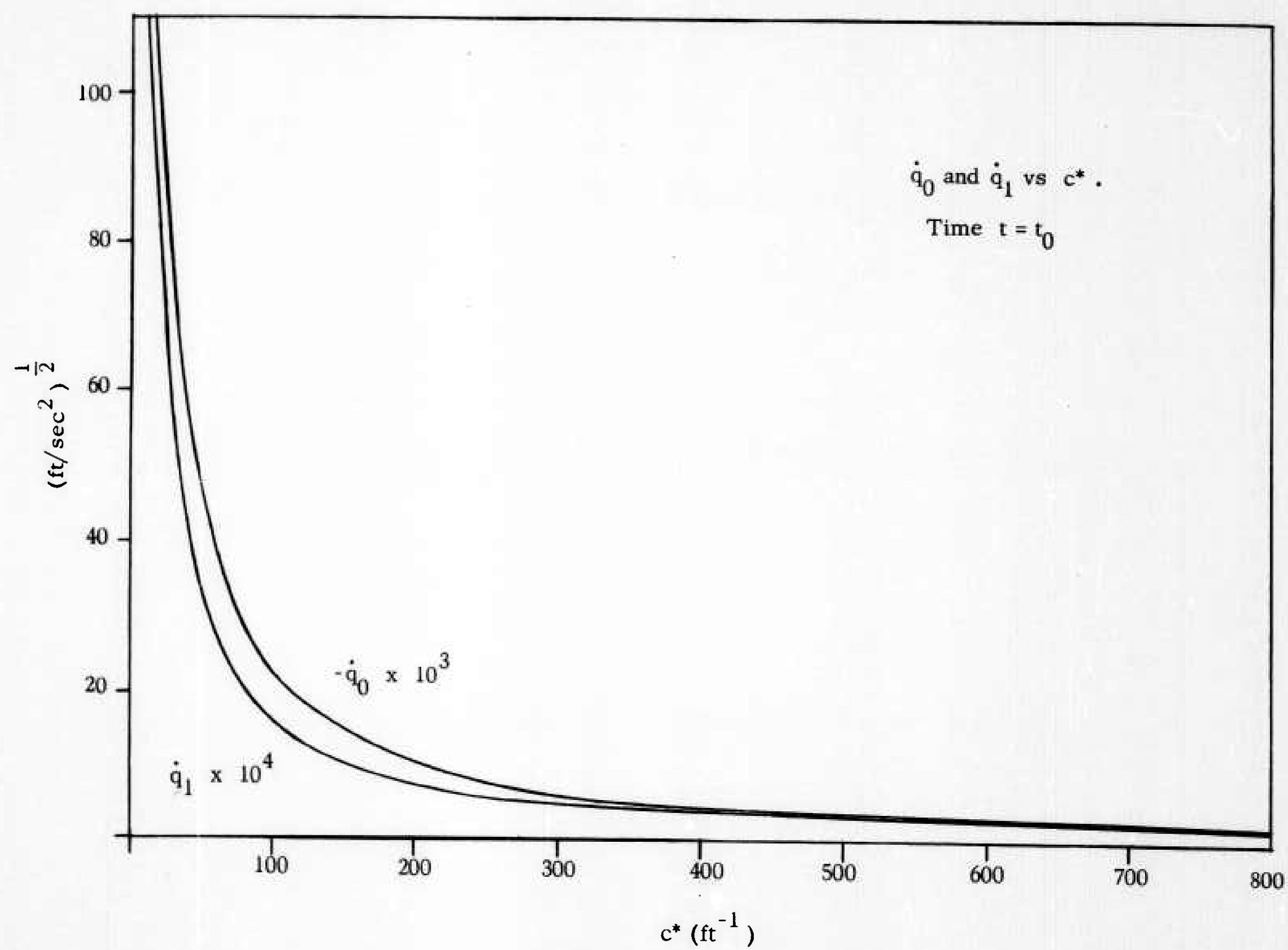

$\dot{q}_0$ and $\dot{q}_1$ vs c^* .
Time $t = t_0$
$-\dot{q}_0 \times 10^3$
$\dot{q}_1 \times 10^4$
$(ft/sec^2)^{\frac{1}{2}}$
100
80
60
40
20
100
200
300
400
500
600
700
800
c^* (ft^{-1})

Figure 13

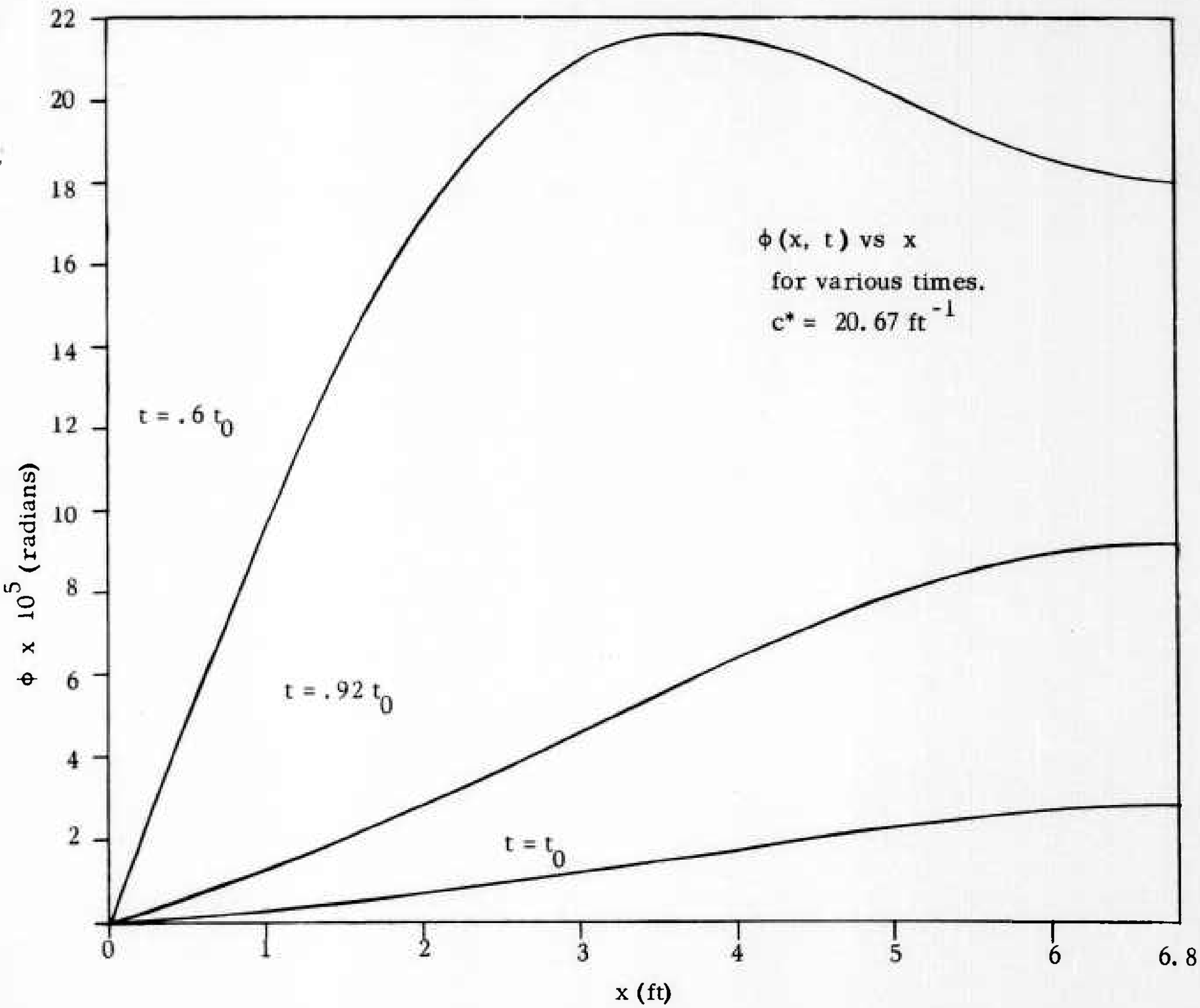

22
20
18
16
14
12
10
8
6
4
2
0
1
2
3
4
5
6
6.8
x (ft)
$\phi \times 10^5$ (radians)
$\phi(x, t)$ vs x
for various times.
$c^* = 20.67\ ft^{-1}$
$t = .6\ t_0$
$t = .92\ t_0$
$t = t_0$

Figure 14

BIBLIOGRAPHY

1. Courant, R. and D. Hilbert, Methods of Mathematical Physics, Vol. 1, Interscience Publishers, Inc., New York, 1953.

2. Lord Rayleigh, The Theory of Sound, Vol. 1, Dover Publications, New York, 1945.

APPENDIX A

In this appendix is presented the analog computer program used in solving Eqs. (61) and (62), which are rewritten here for convenience.

$$\ddot{q}_0 = \frac{\omega_0^2 \beta_1^2}{c^*} q_0 - \omega_0^2 q_0 - \frac{\beta_0 \beta_1 \omega_1^2}{c^*} q_1 + \frac{k \beta_0 \ddot{\xi}}{c^*} \tag{61}$$

$$\ddot{q}_1 = \frac{\omega_1^2 \beta_0^2}{c^*} q_1 - \omega_1^2 q_1 - \frac{\beta_0 \beta_1 \omega_0^2}{c^*} q_0 + \frac{k \beta_1 \ddot{\xi}}{c^*} \tag{62}$$

To increase the computer time to approximately 40 sec., a time scale factor of 4000 was chosen. For the nominal value of c^* equal to 20.67, a preliminary calculation determined that it would be convenient to scale q_0 and q_1 by factors of 10^5 and 5×10^5 respectively. The voltage and time scaled equations then are

$$10^6 q_0'' = \left(\frac{10\, \omega_0^2 \beta_1^2}{a\, c^*}\right) 10^5 q_0 - \left(\frac{10\, \omega_0^2}{a}\right) 10^5 q_0 - \left(\frac{2\, \beta_0\, \beta_1 \omega_1^2}{a\, c^*}\right) 5 \times 10^5 q_1 + \left(\frac{10^3\, k\, \beta_0}{c^*}\right) 10^3 \xi'' \tag{63}$$

and

$$10^6 q_1'' = \left(\frac{2\, \omega_1^2\, \beta_0^2}{a\, c^*}\right) 5 \times 10^5 q_1 - \left(\frac{2\, \omega_1^2}{a}\right) 5 \times 10^5 q_1 - \left(\frac{10\, \omega_0^2\, \beta_0 \beta_1}{a\, c^*}\right) 10^5 q_0 + \left(\frac{10^3\, k\, \beta_1}{c^*}\right) 10^3 \xi'', \tag{64}$$

where $a = 16 \times 10^6$ and the primes denote differentiation with respect to $\tau = 4000\, t$.

The function $\ddot{\xi}$ was chosen to approximate an experimental pressure curve. The analytical expression chosen was

$$\ddot{\xi} = A\, e^{-apt}(1 - \cos pt)\,. \tag{65}$$

where $p = 2\pi \times 10^2$ and $0 \leqq t \leqq .01$. The constant A was determined by requiring that the maximum muzzle velocity be 1550 ft/sec and a was found by requiring that the peak acceleration occur at $.38\, p\, T^*$ where T^* is the period. Utilizing these conditions

$$A = 4.83 \times 10^5 \text{ ft/sec}^2$$

and

$$a = .393\,.$$

Eq. (65) was put into suitable form for programming by integrating twice and eliminating the trigonometric terms. The two constants of integration were determined from the conditions that $\xi(0) = \dot{\xi}(0) = 0$. An appropriate voltage scale factor for this equation was found to be 10^3. The equation which was put on the computer is as follows:

$$10^3 \xi'' = \frac{-ap}{2}\,\xi' - \frac{p^2(1+a^2)\,\xi}{16 \times 10^3} + \frac{A}{16 \times 10^3\, a^2}\left(e^{\frac{-ap\tau}{4000}} - 1\right) + \frac{A\,p\,\tau}{64 \times 10^6\, a}\,. \tag{66}$$

Substituting the numerical values of a, p, and A into Eq. (66), we obtain

$$10^3 \xi'' = -123\,\xi' - 28.5\,\xi + 198.3\,(e^{-.0618\,\tau} - 1) + 12\,\tau\,. \tag{67}$$

The computer block diagram for the solution of Eqs. (63) and (64) is shown in Figure (A.1).

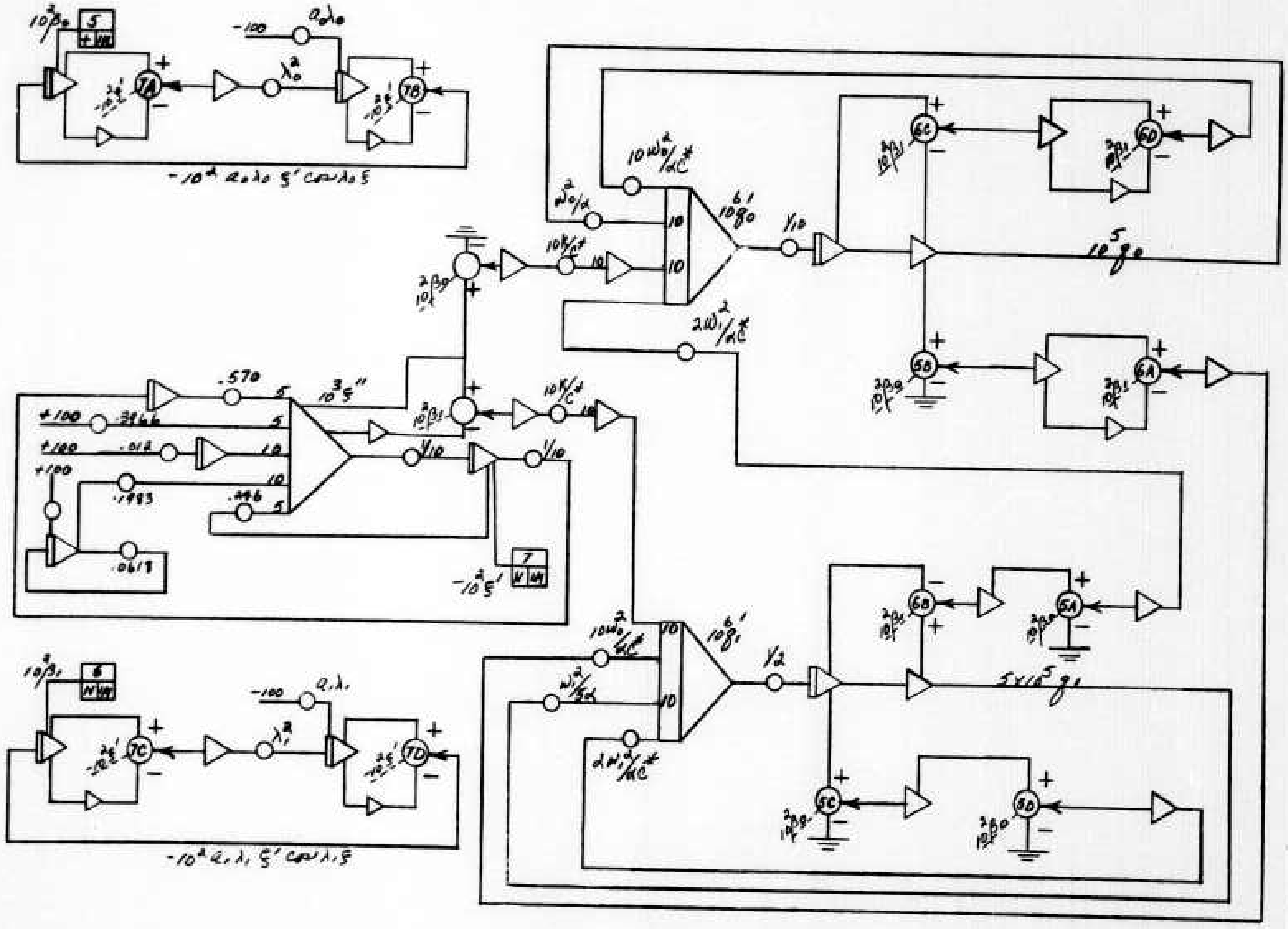


Figure (A. 1)

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Contains an analysis of the torsional oscillations of an artillery barrel when subjected to the time dependent moment of an accelerating projectile. Numerical results are obtained by truncating a modal series which is derived and programming a set of equations on an analog computer.

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